ON THE NUMERICAL THEORY OF SATELLITES WITH HIGHLY INCLINED ORBITS

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ABSTRACT

In this article we develop a numerical lunar theory which can be used to obtain the rectangular coordinates for a satellite moving in a highly inclined orbital plane. The arguments of the theory are the linear functions of the true orbital longitude of the satellite and of the sun and thus they are of Laplacian type. We make use of Hansen's device to perform the integration and for this purpose we introduce a fictitious satellite whose true orbital longitude is considered as a constant till the integration is completed. The perturbations in the orbital plane are obtained by means of a W-function analogous to the W-function of the classical Hansen theory, and the perturbations of the orbital plane carried by the vector $\vec{r}$, the unit vector of the fictitious satellite relative to the mean orbital plane. We show here that the combination of ideas of Laplace, Hansen and Hill represents a convenient way to obtain a numerical lunar theory, at the same time this work also represents a further development and a simplification of the results given by the author in his previous article.

NOTATIONS

$\vec{r}$ - the position vector of the satellite,

$\vec{r}^{o}$ - the unit vector of $\vec{r}$,

$\vec{r}'$ - the position vector of the sun,

$r = | \vec{r} |$,

$u = \frac{1}{r}$,

$\bar{u}$ - the mean value of u,

$\vec{r}'^{o}$ - the unit vector of $\vec{r}'$,

$r' = | \vec{r}' |$,

$u' = \bar{u}' = \frac{1}{r'}$,

$\vec{\Gamma}$ - the unit vector of the fictitious satellite with respect to the system of coordinates associated with the mean orbital plane. This plane is defined here as a plane having the real inclination i but whose longitude of the ascending node is equal to v_3.

$\vec{\bar{\Gamma}}$ - the unit vector of the real satellite with respect to the mean orbital plane.

$1/h$ - the areal velocity of the satellite

h_o - the mean value of h

e - the osculating eccentricity of the satellite

e_o - the mean value of the eccentricity

χ - the true orbital longitude of the pericenter of the satellite

(χ) - the mean value of χ

ϕ - the periodic part of χ

$1-g_1$ - the mean motion of χ

g_2 - the mean motion of the argument of the latitude

g_3 - the mean motion of the ascending node

θ - the longitude of the osculating ascending node

σ - the distance of the departure point from the ascending node

i - the osculating inclination of the orbital plane of the satellite toward the orbital plane of the sun

N' - the periodic periodic part in $-\frac{1}{2}(\theta + \sigma)$

K' - the periodic part in $+\frac{1}{2}(\theta - \sigma)$

v - the true orbital longitude of the satellite

$v_1 = g_1 v - \pi_0$ - the mean true anomaly of the satellite

$v_2 = g_2 v + \omega_0$ - the mean argument of the latitude of the satellite

$v_3 = g_3 v + \theta_0$ - the mean longitude of the ascending node of the satellite

ζ - the pseudo-time (the distrubed time)

$\delta\zeta = \quad - t$ - the perturbations of time

w - the true orbital longitude of the fictitious satellite

$w_1 = g_1 w - \pi_0$ - the mean true anomaly of the fictitious satellite

$w_2 = g_2 w + \omega_0$ - the mean argument of the latitude of the fictitious satellite

m' - the mass of the sun. The orbit of the sun is taken to be elliptic.

Ω - the disturbing function of the satellite. The mass of the planet and the gravitational constant are chosen to be one. The mean of the satellite is supposed to be negligible.

Π - the disturbing function in which the elliptic and the non-elliptic v_2 are separated by replacing in Ω the vector $\bar{\vec{\Gamma}}$ by the vector $\vec{\Gamma}$

W - the Hansen's function determining the perturbations in the orbital plane

$\bar{W} = W\big|_{w_1 = v_1}$

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Introduction

In this article we develop a form of the differential equations of the lunar theory which can be used to obtain the rectangular coordinates for a satellite moving in a highly inclined orbital plane. The integration of these equations is based on solving a certain linear partial differential equation by means of iteration. The author suggested this idea in an earlier work (Musen and al. 1961). The arguments of the theory are the linear functions of the true orbital longitude of the satellite $-v$ and the true orbital longitude v' of the sun and thus they are of Laplacian type. The first suggestion to use v in the theory satellites belongs to Brown (1930). The use of the true orbital longitude speeds up the convergence of the development of the disturbing function, in comparison to the development in terms of the mean anomalies.

As in Hansen's theory, we split the perturbations of the satellite into the perturbations in the orbital plane and into the perturbations of the orbital plane.

We decided to make use of Hansen's device to perform the integration and for this purpose we introduced a fictitious satellite whose true orbital longitude w is considered as a temporary constant, till the integration is completed. After the completion of integration, we apply the "bar operation", replacing w by v, the true orbital longitude of the real satellite.

It is of interest to note that the W-function determining the perturbations in the orbit plane is simpler in the theory of Laplacian type than in the classical Hansen's theory.

Thus the combination of ideas of Laplace, Hansen and Hill represents a convenient way to obtain a numerical theory of the satellite.

The exposition given here represents also a further development and a simplification of the results given by the author in his previous article (1961). In the classical Hansen theory use is made of three auxiliary parameters.

$$P = 2 \sin \tfrac{1}{2} i \sin N'$$

$$Q = 2 \sin \tfrac{1}{2} i \cos N'$$

and K',

where $\mathcal{N}'$ is a purely periodic part in $-(\theta+\sigma)/2$ and K' is a purely periodic part in $+(\theta-\sigma)/2$. For the sake of the symmetry the author (1959) has suggested the use of four parameters

$$\lambda_1 = \sin\tfrac{1}{2} i \cdot \cos \mathcal{N}', \qquad \lambda_3 = \cos\tfrac{1}{2} i \sin K'$$

$$\lambda_2 = \sin\tfrac{1}{2} i \sin \mathcal{N}', \qquad \lambda_4 = \cos\tfrac{1}{2} i \cos K'$$

The implicit role of Hansen's, as well as the author's parameters is only an auxiliary one. They only help to form the components of the unit vector of the satellite with respect to the mean orbit plane. For this reason we discarded their use and resorted to the determination of the unit vector directly.

The differential equation governing the perturbations of the unit vector is simple enough to justify this modification and the total number of the differential equations is reduced for one. The method given here is equally applicable to the planetary satellites disturbed by the sun or to the lunar orbiter disturbed by the Earth, as well as to the artificial satellites of the Earth whose motion is disturbed by the presence of the zonal and the tesseral harmonics. In the first two cases, the eccentricity must be small or moderate and the last case does not include the case of the critical inclination.

Basic Differential Equations

We refer the motion of the satellite to a moving ideal system of coordinates with the x and y axes lying in the osculating orbit plane. The equations of motion can be written in the following standard form:

$$\frac{d^2 r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 = -\frac{1}{r^2} + \frac{\partial \Omega}{\partial r} \tag{1}$$

$$\frac{d}{dt}\left(r^2 \frac{dv}{dt}\right) = \frac{\partial \Omega}{\partial v} \tag{2}$$

Putting

$$r^2 \frac{dv}{dt} = \frac{1}{h} \tag{3}$$

$$u = \frac{1}{r} \tag{4}$$

we can write

$$u = h^2 + h^2 e \cos(v - \chi) \tag{5}$$

The angle χ can be decomposed into the secular part

$$(\chi) = (1 - g_1) v + \pi_0 \tag{6}$$

and into a purely periodic part ϕ . Thus

$$\chi = (1-g_1)v + \pi_0 + \phi \tag{7}$$

From (5) and (7) we have

$$u = h^2 + e h^2 \cos(v_1 - \phi) \tag{8}$$

where we put

$$v_1 = g_1 v - \pi_0 \tag{9}$$

The mean value of u can be defined as

$$\bar{u} = h_0^2 + e_0 h_0^2 \cos V_1 , \tag{10}$$

where e_0 is the constant of the eccentricity and $1/h_0$ is the constant of the "area-integral". Both elements, e_0 and h_0, together with other elements, must be chosen in such a way that no secular or mixed terms appear in the development of the coordinates into trigonometric series.

In order to set Hansen integration procedure it is convenient in the Laplace-Hansen type of lunar theory to define the basic W-function by means of the equation:

$$W = \frac{h}{h_0}\left[1 + e\cos(w_1 - \phi)\right] - \frac{h_0}{h}\left(1 + e_0 \cos w_1\right) , \tag{11}$$

where w_1 is considered as a temporary constant.

Hansen "bar-operation" consists in our case in the replacement of w_1 by V_1 after the integration is completed. In forming the differential equation for W, as well as in the process of integration, w_1 is invariable. This form of the W-function is different from the classical one, but it leads to a simpler differential equation for its determination. The application of the bar-operator to (11) gives:

$$u = \overline{u} + \frac{h}{h_o} h_o^2 \overline{W} \tag{12}$$

Introducing the "stretching factor" $1 + \nu$ by means of the equation

$$\overline{u} = (1+\nu)\, u \tag{12'}$$

we obtain from (12):

$$1 + \nu = \left(1 + \frac{h}{h_o} \cdot \frac{h_o^2}{\overline{u}} \overline{W}\right)^{-1} \tag{13}$$

The practical way of computing ν by means of iteration is based on the use of formula:

$$\nu = -(1+\nu) \frac{h}{h_o} \cdot \frac{h_o^2}{\overline{u}} \overline{W} . \tag{13'}$$

Making use of the equation (Brown, 1896)

$$\frac{d}{dt}\left[\frac{h}{h_0} e \cos(\chi-\beta)\right] = \frac{1}{h_0}\,\frac{\partial\Omega}{\partial r}\sin(v-\beta) \tag{14}$$

$$+ h_0 \frac{\partial\Omega}{\partial v}\left(\frac{u}{h_0^2}+\frac{h^2}{h_0^2}\right)\cos(v-\beta)$$

$$+ \frac{h}{h_0}\, e\, \frac{d\beta}{dt}\sin(\chi-\beta)$$

and putting

$$\beta = v - v_1 + w_1$$

we obtain

$$\frac{d}{dt}\left[\frac{h}{h_0} e \cos(w_1-\phi)\right] = \frac{1}{h_0}\,\frac{\partial\Omega}{\partial r}\sin(v_1-w_1) \tag{15}$$

$$+ h_0 \frac{\partial\Omega}{\partial v}\left(\frac{u}{h_0^2}+\frac{h^2}{h_0^2}\right)\cos(v_1-w_1)$$

$$+ (1-g_1)\frac{h}{h_0}\, e\, \frac{dv}{dt}\sin(\phi-w_1)$$

Combining the equation (15) with the standard equations

$$\frac{d}{dt}\,\frac{h}{h_o} = -\frac{h^2}{h_o^2}\cdot h_o\,\frac{\partial\Omega}{\partial v}$$

$$\frac{d}{dt}\,\frac{h_o}{h} = \qquad + h_o\,\frac{\partial\Omega}{\partial v}$$

together, we deduce :

$$\frac{dW}{dt} = \Big[\Big(\frac{h^2}{h_o^2} + \frac{u}{h_o^2}\Big)\cos(v_1 - w_1) - \Big(\frac{h^2}{h_o^2} + 1 + e_o\cos w_1\Big)h_o\frac{\partial\Omega}{\partial v} \tag{16}$$

$$+ \frac{1}{h_o}\frac{\partial\Omega}{\partial r}\sin(v_1 - w_1) + \frac{h}{h_o}\,e\,(1-g_1)\,\frac{dv}{dt}\sin(\phi - w_1)$$

Taking (12) into account, we obtain :

$$\frac{dW}{dt} = \Big[\Big(\frac{h^2}{h_o^2} + \frac{\overline{u}}{h_o^2} + \frac{h}{h_o}\,\overline{W}\Big)\cos(v_1 - w_1) \tag{17}$$

$$-\Big(\frac{h^2}{h_o^2} + 1 + e_o\cos w_1\Big)\Big]\,h_o\,\frac{\partial\Omega}{\partial v}$$

$$+ \frac{1}{h_o}\frac{\partial\Omega}{\partial r}\sin(v_1 - w_1) + \frac{h}{h_o}\,e\,(1-g_1)\,\frac{dv}{dt}\sin(\phi - w_1)$$

Eliminating sin $(\phi - w_1)$ from the last equation by means of the relation

$$\frac{h}{h_0} e \sin(\phi - w_1) = \frac{\partial W}{\partial w_1} - \frac{h_0}{h} e_0 \sin w_1 \tag{18}$$

and taking

$$\frac{dt}{dv} = \frac{h_0}{u^2}$$

into consideration, we have

$$\frac{dW}{dv} = N \frac{\partial \Omega}{\partial u} + M \frac{h_0^2}{u^2} \frac{\partial \Omega}{\partial v} + (1 - g_1)\left(\frac{\partial W}{\partial w_1} - \frac{h_0}{h} e_0 \sin w_1\right) \tag{19}$$

where we put

$$N = -\frac{h}{h_0}\sin(v_1 - w_1),$$

$$M = \frac{h}{h_0}\left[\left(\frac{h^2}{h_0^2} + 1 + e_0\cos w_1 + \frac{h}{h_0}\overline{W}\right)\cos(v_1 - w_1)\right.$$

$$\left. - \left(\frac{h^2}{h_0^2} + 1 + e_0\cos w_1\right)\right]$$

The motion of the pericenter $1 - g_1$ is obtained from the condition that no term of the form $A\cos w_1$ appears in the equation (19).

If the eccentricity e_0 is not very small, say, of the order 0.01-0.3, approximately, then the numerical values of all the elements can be substituted from the outset.

If the eccentricity is smaller than, approximately, 0.01, then in order to avoid the numerical difficulty in the determination of the motion of the perigee, it is recommended to keep the first power of the eccentricity in the literal form. In other words, every trigonometrical series T must be written in the form

$$T = A_0 + e_0 A_1$$

where A_0 is independent from e_0 and both series, A_0 and A_1, are the trigonometric series with purely numerical coefficients. We write the product of two such series in the form

$$(A_0 + e_0 A_1)(B_0 + e_0 B_1) = A_0 B_0 + e_0 (A_1 B_0 + A_0 B_1 + e_0 A_1 B_1)$$

and, again, the numerical value of e_0 is substituted in the parentheses. Such a simple device serves as a safeguard against the "small divisor" in the determination of the motion of the pericenter from equation (19). The unit vector $\vec{r}^{\,0}$ can written in the form

$$\vec{r}^{\,0} = A_3(\theta) \cdot A_1(i) \cdot A_3(-\sigma) \cdot \begin{bmatrix} \cos v \\ \sin v \\ 0 \end{bmatrix} \tag{20}$$

where

$$A_1(\alpha) = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos\alpha & -\sin\alpha \\ 0 & +\sin\alpha & +\cos\alpha \end{bmatrix}$$

and

$$A_3(\alpha) = \begin{bmatrix} +\cos\alpha & -\sin\alpha & 0 \\ +\sin\alpha & +\cos\alpha & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

and we have

$$\vec{r}^{\,\circ} = A_3(\theta) \cdot A_1(i) \cdot \begin{bmatrix} \cos(v - \sigma) \\ \sin(v - \sigma) \\ 0 \end{bmatrix}$$

we can set

$$v - \sigma = v_2 + N' + K'$$

$$\theta = v_3 - N' + K'$$

where v_2 and v_3 are the linear arguments with respect to v,

$$v_2 = g_2 v + \omega_0$$

$$v_3 = g_3 v + \theta_0$$

and $\mathcal{N}'$ and K' are purely periodic. The argument v_2 represents the mean argument of the latitude and v_3 represents the mean longitude of the ascending node. The coefficient g_2 is of order of one and g_3 is of the order of perturbations.

Using the auxiliary parameters

$$\lambda_1 = \sin\tfrac{1}{2}i \cos \mathcal{N}', \qquad \lambda_3 = -\cos\tfrac{1}{2}i \sin K',$$

$$\lambda_2 = \sin\tfrac{1}{2}i \sin \mathcal{N}', \qquad \lambda_4 = \cos\tfrac{1}{2}i \cos K',$$

the unit vector $\vec{r}^{\,\circ}$ can be represented as (Musen, 1961):

$$\vec{r}^{\,\circ} = A_3(v_3) \cdot \Lambda \cdot \begin{bmatrix} \cos v_2 \\ \sin v_2 \\ 0 \end{bmatrix}, \tag{21}$$

where the matrix Λ carries all the periodic effects in
The elements λ_{ij} of the matrix Λ are simple polynomials in

$\lambda_1, \lambda_2, \lambda_3, \lambda_4$:

$$\begin{aligned} \lambda_{11} &= +\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2, \\ \lambda_{21} &= +2\lambda_3\lambda_4 - 2\lambda_1\lambda_2, \\ \lambda_{31} &= +2\lambda_3\lambda_1 + 2\lambda_2\lambda_4, \end{aligned} \tag{22}$$

$$
\begin{aligned}
\lambda_{12} &= -2\lambda_3\lambda_4 - 2\lambda_1\lambda_2 \\
\lambda_{22} &= -\lambda_1^2 + \lambda_2^2 - \lambda_3^2 + \lambda_4^2 \\
\lambda_{32} &= +2\lambda_1\lambda_4 - 2\lambda_2\lambda_3
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\lambda_{13} &= +2\lambda_1\lambda_3 - 2\lambda_2\lambda_4 \\
\lambda_{23} &= -2\lambda_1\lambda_4 - 2\lambda_2\lambda_3 \\
\lambda_{33} &= -\lambda_1^2 - \lambda_2^2 + \lambda_3^2 + \lambda_4^2
\end{aligned}
\tag{24}
$$

The author has established (Musen, 1963) the following set of the differential equations for the variations of the λ - parameters:

$$
\begin{aligned}
\frac{d\lambda_1}{dt} = &-\frac{1}{2}(1 - g_2 + g_3)\frac{dv}{dt}\lambda_2 \\
&+ \frac{1}{4}h\Big[+(\lambda_4^2 + \lambda_3^2)\frac{\partial\Omega}{\partial\lambda_2} - (\lambda_1\lambda_4 + \lambda_2\lambda_3)\frac{\partial\Omega}{\partial\lambda_3} \\
&- (\lambda_2\lambda_4 - \lambda_1\lambda_3)\frac{\partial\Omega}{\partial\lambda_4}\Big]
\end{aligned}
\tag{25}
$$

$$\frac{d\lambda_2}{dt} = +\frac{1}{2}(1-g_2+g_3)\frac{dV}{dt}\lambda_1 \tag{26}$$

$$+\frac{1}{4}h\Big[-(\lambda_4^2+\lambda_3^2)\frac{\partial\Omega}{\partial\lambda_1}-(\lambda_2\lambda_4-\lambda_1\lambda_3)\frac{\partial\Omega}{\partial\lambda_3}$$

$$+(\lambda_1\lambda_4+\lambda_2\lambda_3)\frac{\partial\Omega}{\partial\lambda_4}\Big]$$

$$\frac{d\lambda_3}{dt} = +\frac{1}{2}(1-g_2-g_3)\frac{dV}{dt}\lambda_4 \tag{27}$$

$$+\frac{1}{4}h\Big[-(\lambda_1^2+\lambda_2^2)\frac{\partial\Omega}{\partial\lambda_4}+(\lambda_1\lambda_4+\lambda_2\lambda_3)\frac{\partial\Omega}{\partial\lambda_1}$$

$$+(\lambda_2\lambda_4-\lambda_1\lambda_3)\frac{\partial\Omega}{\partial\lambda_2}\Big]$$

$$\frac{d\lambda_4}{dt} = -\frac{1}{2}(1-g_2-g_3)\frac{dV}{dt}\lambda_3 \tag{28}$$

$$+\frac{1}{4}h\Big[+(\lambda_1^2+\lambda_2^2)\frac{\partial\Omega}{\partial\lambda_3}+(\lambda_2\lambda_4-\lambda_1\lambda_3)\frac{\partial\Omega}{\partial\lambda_1}$$

$$-(\lambda_1\lambda_4+\lambda_2\lambda_3)\frac{\partial\Omega}{\partial\lambda_2}\Big]$$

From equations (22) - (28) we deduce:

$$\frac{d\lambda_{11}}{dt} = \left[+(1-g_2)\,\lambda_{12} + g_3\,\lambda_{21}\right]\frac{dv}{dt} + \frac{1}{2}h\,A\,\lambda_{13} \tag{30}$$

$$\frac{d\lambda_{12}}{dt} = \left[-(1-g_2)\,\lambda_{11} + g_3\,\lambda_{22}\right]\frac{dv}{dt} + \frac{1}{2}h\,B\,\lambda_{13} \tag{31}$$

$$\frac{d\lambda_{21}}{dt} = \left[+(1-g_2)\,\lambda_{22} - g_3\,\lambda_{11}\right]\frac{dv}{dt} + \frac{1}{2}h\,A\,\lambda_{23} \tag{32}$$

$$\frac{d\lambda_{22}}{dt} = \left[-(1-g_2)\,\lambda_{21} - g_3\,\lambda_{12}\right]\frac{dv}{dt} + \frac{1}{2}h\,B\,\lambda_{23} \tag{33}$$

$$\frac{d\lambda_{31}}{dt} = +(1-q_2)\,\lambda_{32}\,\frac{dv}{dt} + \frac{1}{2}\,h\,A\,\lambda_{33} \tag{34}$$

$$\frac{d\lambda_{32}}{dt} = -(1-q_2)\,\lambda_{31}\,\frac{dv}{dt} + \frac{1}{2}\,h\,B\,\lambda_{33} \tag{35}$$

where we put :

$$A = -\lambda_4\frac{\partial\Omega}{\partial\lambda_1} + \lambda_3\frac{\partial\Omega}{\partial\lambda_2} - \lambda_2\frac{\partial\Omega}{\partial\lambda_3} + \lambda_1\frac{\partial\Omega}{\partial\lambda_4} \tag{36}$$

$$B = +\lambda_3\frac{\partial\Omega}{\partial\lambda_1} + \lambda_4\frac{\partial\Omega}{\partial\lambda_2} - \lambda_1\frac{\partial\Omega}{\partial\lambda_3} - \lambda_2\frac{\partial\Omega}{\partial\lambda_4} \tag{37}$$

The simplicity of equations (30)-(35), as well as equation (21), show that we can discard the use of λ - parameters and to introduce the unit vector

$$\vec{\Gamma} = \Gamma_1 \vec{i} + \Gamma_2 \vec{j} + \Gamma_3 \vec{k}, \tag{38}$$

$$\Gamma_1 = \lambda_{11} \cos W_2 + \lambda_{12} \sin W_2, \tag{39}$$

$$\Gamma_2 = \lambda_{21} \cos W_2 + \lambda_{22} \sin W_2, \tag{40}$$

$$\Gamma_3 = \lambda_{31} \cos W_2 + \lambda_{32} \sin W_2, \tag{41}$$

$$\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = 1$$

instead. We shall consider w_2 constant during the formation of the differential equation for $\vec{r}$ and during the integration. After the integration is completed, we apply the bar-operation to and replace w_2 with v_2. The vector $\vec{\bar{r}}$ is the unit vector of the satellite relative to the mean orbital plane.

Thus

$$\vec{\bar{r}} = \bar{r}_1 \vec{i} + \bar{r}_2 \vec{j} + \bar{r}_3 \vec{k} \tag{38'}$$

$$\bar{r}_1 = \lambda_{11} \cos v_2 + \lambda_{12} \sin v_2 , \tag{39'}$$

$$\bar{r}_2 = \lambda_{21} \cos v_2 + \lambda_{22} \sin v_2 , \tag{40'}$$

$$\overline{\Gamma}_3 = \lambda_{31} \cos v_2 + \lambda_{32} \sin v_2 , \tag{41'}$$

$$\overline{\Gamma}_1^2 + \overline{\Gamma}_2^2 + \overline{\Gamma}_3^2 = 1 ,$$

and equation (21) takes a concise form

$$\vec{r}^{\,\circ} = A_3 (v_3) \cdot \vec{\overline{\Gamma}} : \tag{42}$$

In performing iteration we have to distinguish between v_2 in the development of perturbations and the "elliptic" v_2 in the vector $\vec{\overline{\Gamma}}$.

The derivatives of the disturbing function are formed with respect to the "elliptic" v_2. For this reason it is convenient to separate the elliptic and the non-elliptic v_2 in the disturbing function by temporarily replacing the vector $\vec{\bar{r}}$ with $\vec{r}$.

The disturbing function so modified will be designated by $\overline{\Pi}$. Taking (39)-(41) into account, we obtain

$$\frac{1}{2}\left(-\lambda_4\frac{\partial\overline{\Pi}}{\partial\lambda_1}+\lambda_3\frac{\partial\overline{\Pi}}{\partial\lambda_2}-\lambda_2\frac{\partial\overline{\Pi}}{\partial\lambda_3}+\lambda_1\frac{\partial\overline{\Pi}}{\partial\lambda_4}\right)$$

$$=-\left(\frac{\partial\overline{\Pi}}{\partial r_1}\lambda_{13}+\frac{\partial\overline{\Pi}}{\partial r_2}\lambda_{23}+\frac{\partial\overline{\Pi}}{\partial r_3}\lambda_{33}\right)\sin w_2$$

$$=-\vec{R}\cdot\nabla_{\vec{\bar{r}}}\overline{\Pi}\sin w_2,$$

$$\frac{1}{2}\left(+\lambda_3\frac{\partial\overline{\Pi}}{\partial\lambda_1}+\lambda_4\frac{\partial\overline{\Pi}}{\partial\lambda_2}-\lambda_1\frac{\partial\overline{\Pi}}{\partial\lambda_3}-\lambda_2\frac{\partial\overline{\Pi}}{\partial\lambda_4}\right)$$

$$=+\left(\frac{\partial\overline{\Pi}}{\partial r_1}\lambda_{13}+\frac{\partial\overline{\Pi}}{\partial r_2}\lambda_{23}+\frac{\partial\overline{\Pi}}{\partial r_3}\lambda_{33}\right)\cos w_2$$

$$=+\vec{R}\cdot\nabla_{\vec{\bar{r}}}\overline{\Pi}\cos w_2,$$

and

$$\frac{1}{2}A=-C\sin v_2$$

$$\frac{1}{2}B=+C\cos v_2$$

where we put

$$C=\overline{\frac{\partial\Pi}{\partial r_1}}\lambda_{13}+\overline{\frac{\partial\Pi}{\partial r_2}}\lambda_{23}+\overline{\frac{\partial\Pi}{\partial r_3}}\lambda_{33}$$

$$\frac{d\Gamma_2}{dV} = +(1-g_2)\frac{\partial \Gamma_2}{\partial W_2} - g_3\Gamma_1 + \frac{h^2}{h_0^2}\cdot\frac{h_0^2}{u^2}\, C\, \lambda_{23}\sin(W_2-V_2) \quad (44)$$

$$\frac{d\Gamma_3}{dV} = +(1-g_2)\frac{\partial \Gamma_3}{\partial W_2} \qquad + \frac{h^2}{h_0^2}\cdot\frac{h_0^2}{u^2}\, C\, \lambda_{33}\sin(W_2-V_2) \quad (45)$$

or, in the vectorial form:

$$\frac{d\vec{\Gamma}}{dV} = +(1-g_2)\frac{\partial \vec{\Gamma}}{\partial W_2} + g_3\,\vec{\Gamma}\times\vec{k} + \frac{h^2}{h_0^2}\cdot\frac{h_0^2}{u^2}\,\vec{R}\, C\,\sin(W_2-V_2) \quad (46)$$

The components $\lambda_{13}, \lambda_{23}, \lambda_{33}$ of the vector $\vec{R}$ are computed using the formula

$$\vec{R} = \left(\vec{r} \times \frac{\partial \vec{r}}{\partial w_2} \right)_{w_2 = 0}$$

for the lunar case we obtain, taking (42) into account:

$$\cos H = \vec{r}^{\,\circ} \cdot \vec{r}'^{\,\circ} = \left[\cos V_4 , \sin V_4 , 0 \right] \cdot A_3 (V_3) \cdot \vec{\bar{r}} \tag{47}$$

$$= \bar{r}_1 \cos (V_3 - V_4) - \bar{r}_2 \sin (V_3 - V_4)$$

Thus, in performing the transition from Ω to Π we must substitute the following expression for cos H:

$$r_1 \cos (V_3 - V_4) - r_2 \sin (V_3 - V_4) .$$

From

$$\Omega = \frac{m' r^2}{r'^3} P_2 (\cos H) + \frac{m' r^3}{r'^4} P_3 (\cos H)$$

$$+ \frac{m' r^4}{r'^5} P_4 (\cos H) + \ldots\ldots$$

we obtain:

$$\Pi = \lambda^2 h_0^2 \,(1+\nu)^2 \left(\frac{\bar{u}'}{\lambda'^2}\right)^3 \left(\frac{h_0^2}{\bar{u}}\right)^2 \Big[\left(+\frac{3}{4}\Gamma_1^2 + \frac{3}{4}\Gamma_2^2 - \frac{1}{2}\right)$$

$$+ \frac{3}{4}\left(\Gamma_1^2 - \Gamma_2^2\right) \cos(2V_3 - 2V_4) - \frac{3}{2}\Gamma_1\Gamma_2 \sin(2V_3 - 2V_4)\Big]$$

$$+ \lambda^2 h_0^2 \,\alpha\,(1+\nu)^3 \left(\frac{\bar{u}'}{h'^2}\right)^4 \left(\frac{h_0^2}{\bar{u}}\right)^3 \Big[\left(+\frac{15}{8}\Gamma_1^3 + \frac{15}{8}\Gamma_1\Gamma_2^2 - \frac{3}{2}\Gamma_1\right)\cos(V_3 - V_4)$$

$$+ \left(-\frac{15}{8}\Gamma_1^2\Gamma_2 - \frac{15}{8}\Gamma_2^3 + \frac{3}{2}\Gamma_2\right)\sin(V_3 - V_4)$$

$$+ \left(+\frac{5}{8}\Gamma_1^3 - \frac{15}{8}\Gamma_1\Gamma_2^2\right)\cos(3V_3 - 3V_4)$$

$$+ \left(-\frac{15}{8}\Gamma_1^2\Gamma_2 + \frac{5}{8}\Gamma_2^3\right)\sin(3V_3 - 3V_4)$$

$$+ \lambda^2 h_0^2 \,\alpha^2 (1+\nu)^4 \left(\frac{\bar{u}'}{h'^2}\right)^4 \left(\frac{h_0^2}{\bar{u}}\right)^3 \Big[\Big(+\frac{105}{64}\Gamma_1^4 + \frac{105}{32}\Gamma_1^2\Gamma_2^2$$

$$+ \frac{105}{64}\Gamma_2^4 - \frac{15}{8}\Gamma_1^2 - \frac{15}{8}\Gamma_2^2 + \frac{3}{8}\Big)$$

$$+ \left(+\frac{35}{16}\Gamma_1^4 - \frac{35}{16}\Gamma_2^4 - \frac{15}{8}\Gamma_1^2 + \frac{15}{8}\Gamma_2^2\right)\cos(2V_3 - 2V_4)$$

$$+ \left(-\frac{35}{8}\Gamma_1^3\Gamma_2 - \frac{35}{8}\Gamma_1\Gamma_2^3 + \frac{15}{8}\Gamma_1\Gamma_2\right)\sin(2V_3 - 2V_4)$$

$$+ \left(+\frac{35}{64}\Gamma_1^4 + \frac{35}{64}\Gamma_2^4\right)\cos(4V_3 - 4V_4)$$

$$+ \left(-\frac{35}{16}\Gamma_1^3\Gamma_2 + \frac{35}{16}\Gamma_1\Gamma_2^3\right)\sin(4V_3 - 4V_4)\Big] + \cdots\cdots$$

where

$$\lambda^2 = m' \left(\frac{h'}{h_0}\right)^6,$$

$$\alpha = \left(\frac{h'}{h_0}\right)^2,$$

λ is the analogue of the parameter m of Delaunay's theory and α is the parallactic factor of our theory. As in (43)-(45) equation (19) becomes:

$$\frac{dW}{dv} = N \overline{\frac{\partial \Pi}{\partial u}} + M \frac{h_0^2}{u^2} \overline{\frac{\partial \Pi}{\partial w_2}} + (1 - g_2)\left(\frac{\partial W}{\partial w_1} - \frac{h_0}{h} e_0 \sin w_1\right) \tag{19'}$$

Equations (43)-(45) and (19') can be simplified in the case of the artificial satellite, providing only zonal harmonics are considered. We have

$$\Pi = k_2 u^3 (1 - 3\Gamma_3^2) + k_3 u^4 (3\Gamma_3 - 5\Gamma_3^2) + k_4 u^5 (3 - 30\Gamma_3^2 + 35\Gamma_3^4) + \ldots$$

$$\frac{d\Gamma_1}{dV} = +(1-g_2)\frac{\partial \Gamma_1}{\partial w_2} + g_3\,\Gamma_2$$

$$+ \frac{h^2}{h_0^2}\cdot\frac{h_0^2}{u^2}\,\overline{\frac{\partial \Pi}{\partial \Gamma_3}}\,\lambda_{33}\,\lambda_{13}\sin(w_2 - v_2),$$

$$\frac{d\Gamma_2}{dV} = +(1-g_2)\frac{\partial \Gamma_2}{\partial w_2} - g_3\,\Gamma_1$$

$$+ \frac{h^2}{h_0^2}\cdot\frac{h_0^2}{u^2}\,\overline{\frac{\partial \Pi}{\partial \Gamma_3}}\,\lambda_{33}\,\lambda_{23}\sin(w_2 - v_2),$$

$$\frac{d\Gamma_3}{dV} = +(1-g_2)\frac{\partial \Gamma_3}{\partial w_2}$$

$$+ \frac{h^2}{h_0^2}\cdot\frac{h_0^2}{u^2}\,\overline{\frac{\partial \Pi}{\partial \Gamma_3}}\,\lambda_{33}^2\sin(w_2 - v_2),$$

$$\frac{dW}{dV} = N\,\overline{\frac{\partial \Pi}{\partial u}} + M\,\frac{h_0^2}{u^2}\,\overline{\frac{\partial \Pi}{\partial \Gamma_3}}\cdot\overline{\frac{\partial \Gamma_3}{\partial w_2}}$$

$$+ (1-g_2)\left(\frac{\partial W}{\partial w_1} - e_0\frac{h_0}{h}\sin w_1\right).$$

We start iteration with

$$\frac{h}{h_0} = 1 \quad , \qquad 1 + \nu = 1,$$

$$\Gamma_1 = \cos w_2 \quad , \quad \Gamma_2 = \cos i_0 \sin w_2 \ , \quad \Gamma_3 = \sin i_0 \sin w_2$$

and repeat it till we reach the final values. The perturbed coordinates are the trigonometric series in four arguments:

$$v_1 \ , \ v_2 , \ v_3 , \ v_4$$

In the classical Laplacian theory the angle v_4 is eliminated in favor of the angle $mv-c_4$, where c_4 is a constant. We can avoid this elimination, as well as the substitution of trigonometric series into the arguments, if we base the solution on integration of a partial differential equation instead of performing the quadratures.

We have:

$$\frac{dv_4}{dv} = \frac{h}{h'} \left(\frac{u'}{u}\right)^2 \sqrt{1+m'} \quad ,$$

or

$$\frac{dv_4}{dv} = \sqrt{1+m'} \quad \left(\frac{h'}{h_0}\right)^3 (1+\nu)^2 \frac{h}{h_0} \left(\frac{\bar{u}'/h'^2}{\bar{u}/h_0^2}\right)^2 . \tag{48}$$

Designating the constant part in the right side of the last equation by g_4, we have :

$$\frac{d}{dv} = g_1 \frac{\partial}{\partial v_1} + g_2 \frac{\partial}{\partial v_2} + g_3 \frac{\partial}{\partial v_3} + g_4 \frac{\partial}{\partial v_4} + K \frac{\partial}{\partial v_4},$$

where we put :

$$K = \left(\frac{h'}{h_0}\right)^3 \cdot \frac{h}{h_0} (1+\nu)^2 \left(\frac{\bar{u}'/l'^2}{\bar{u}/l_0^2}\right)^2 - g_4 , \tag{49}$$

and K is a purely periodic series.

Thus, we reduce the problem to integration of the partial differential equations:

$$g_1 \frac{\partial W}{\partial v_1} + g_2 \frac{\partial W}{\partial v_2} + g_3 \frac{\partial W}{\partial v_3} + g_4 \frac{\partial W}{\partial v_4} - \tag{19''}$$

$$= N \overline{\frac{\partial \Pi}{\partial u}} + M \frac{h_0^2}{u^2} \overline{\frac{\partial \Pi}{\partial w_2}} - K \frac{\partial W}{\partial v_4}$$

$$+ (1-g_2) \left(\frac{\partial W}{\partial w_1} - \frac{h_0}{h} e_0 \sin w_1\right)$$

$$g_1 \frac{\partial \vec{r}}{\partial v_1} + g_2 \frac{\partial \vec{r}}{\partial v_2} + g_3 \frac{\partial \vec{r}}{\partial v_3} + g_4 \frac{\partial \vec{r}}{\partial v_4} \tag{46'}$$

$$= (1 - g_2) \frac{\partial \vec{r}}{\partial w_2} + g_3 \, \vec{r} \times \vec{k}$$

$$+ \frac{h^2}{h_0^2} \cdot \frac{h_0^2}{u^2} \vec{R} \, C \sin (w_2 - v_2) - K \frac{\partial \vec{r}}{\partial v_4},$$

and the right sides of these equations must not contain constant terms.

From (49) it can be seen easily that

$$K = O(e_0 \lambda)$$

and it is a small quantity. The terms

$$K \frac{\partial W}{\partial v_4}, \quad K \frac{\partial \vec{r}}{\partial v_4}$$

numerically are at least of the fourth order in λ . Consequently, the solution of the partial differential equations (19") and (46') by means of iteration represents a fast convergent process.

Some simplifications are possible in the case of an artificial satellite. If we include the effects of the tesseral harmonics, then the argument v_4 is the sidereal time on the zero-meridian. Designating the angular velocity of rotation of the Earth by n', we have

$$\frac{dv_4}{dv} = \frac{n'}{k_0^3} \cdot \frac{k}{k_0} \left(\frac{k_0^2}{u} \right)^2 .$$

Let g_4 be the constant part in the right side of the last equation and put:

$$K = \frac{n'}{k_0^3} \cdot \frac{k}{k_0} \cdot \left(\frac{k_0^2}{u} \right)^2 - g_4$$

We have:

$$\frac{d}{dv} = g_1 \frac{\partial}{\partial v_1} + g_2 \frac{\partial}{\partial v_2} + (g_3 - g_4) \frac{\partial}{\partial v_3} - K \frac{\partial}{\partial v_3} ,$$

because the arguments v_3 and v_4 appear in the form of the difference

$v_3 - v_4$.

Evidently

$$K = O\left(\frac{n'}{\ell_o^3} e_o\right)$$

If we consider the case of geodetic satellites with small eccentricities, then the quantity $n' e_o / \ell_o^3$ is small enough to justify the use of iteration. For example, for the satellite with

$$a = 1.87 R_\oplus \quad , \quad e = 0.01$$

the factor $n' e_o / \ell_o^3$ is of the order 10^{-3}.

If only the zonal harmonics are considered, then

$$\frac{d}{dv} = g_1 \frac{\partial}{\partial v_1} + g_2 \frac{\partial}{\partial v_2}$$

and equations (19") and (46') become

$$g_1 \frac{\partial W}{\partial v_1} + g_2 \frac{\partial W}{\partial v_2} = N \overline{\frac{\partial \Pi}{\partial u}} + M \frac{\ell_o^2}{u^2} \overline{\frac{\partial \Pi}{\partial w_2}}$$

$$+ (1 - g_2)\left(\frac{\partial W}{\partial w_1} - \frac{\ell_o}{\ell} e_o \sin w_1\right),$$

$$g_1 \frac{\partial \vec{r}}{\partial v_1} + g_2 \frac{\partial \vec{r}}{\partial v_2} = (1 - g_2) \frac{\partial \vec{r}}{\partial w_2} + g_3 \vec{r} \times \vec{k}$$
$$+ \frac{h^2}{h_0^2} \cdot \frac{h_0^2}{u^2} \vec{R}\, C \sin (w_2 - v_2)$$

For the time being the author's preference is on the side of numerical theories. Taking into account the speed and accuracy with which modern machines perform the operations, it is evident that the position of purely numerical theories is very satisfactory. However, the way to solve equations (19') and (46') by an analytical method is not closed either. We can follow the way suggested by Poincare (1892) and represent the frequencies g_1, g_2, g_3, g_4 and W and $\vec{r}$ as power series Then we develop the operator

$$g_1 \frac{\partial}{\partial v_1} + g_2 \frac{\partial}{\partial v_2} + g_3 \frac{\partial}{\partial v_3} + g_4 \frac{\partial}{\partial v_4}$$

into a power series in The coefficients in this series are also operators. Then we are able to dissolve equations (19') and (46') into equations integrable by quadratures.

Constants of Integration

Determining integration constants in this theory is simpler than in the author's previous theory (1959).

In the lunar case, the series for $\lambda_{11}, \lambda_{22}, \lambda_{32}$ are the cosine series and, consequently, they contain the additive constant of integration. The series for $\lambda_{12}, \lambda_{21}, \lambda_{31}$ are the sine series and they do not contain any such constants. The same conclusions are valid also for artificial satellites. We conclude that the form of the additive constant of integration in Γ_1 is

$$(1 + c_1') \cos w_2$$

in Γ_2 it is

$$(\cos i_0 + c_2') \sin w_2$$

and in Γ_3 it is

$$(\sin i_0 + c_3') \sin w_2$$

We shall determine c_2 and c_3 in such a way that $\cos i_0 \sin v_2$

will be the only term in Γ_2 which is of the zero order and $\sin i_0 \sin v_2$ will be the only term of zero order in Γ_3.

The series for Γ_1, will be the cosine series in v_1, v_2, v_3, v_4 and Γ_2, Γ_3 will be the sine series in these arguments.

We conclude, that the term in

$$+(1-g_2)\frac{\partial \Gamma_1}{\partial w_2} + g_3 \Gamma_2$$

independent from v_1, v_2, v_3, v_4 must contain $\sin w_2$ as a factor.

In a similar way, the terms in

$$+(1-g_2)\frac{\partial \Gamma_2}{\partial w_2} - g_3 \Gamma_1 .$$

and in

$$+(1-g_2)\frac{\partial \Gamma_3}{\partial w_2}$$

which are independent from v_1, v_2, v_3, v_4 must have $\cos w_2$ as a factor. In order to avoid the secular terms in the components of $\vec{\Gamma}$ we must remove the terms containing the argument w_2 alone from the derivatives of $\vec{\Gamma}$. Let K_1 be the coefficient of $\sin w_2$ in

$$\frac{h^2}{h_0^2} \cdot \frac{h_0^2}{u^2} \, C \, \lambda_{13} \sin(w_2 - v_2) ,$$

K_2 be the coefficient of cos w_2 in

$$\frac{h^2}{h_0^2} \cdot \frac{h_0^2}{u^2} \, C \, \lambda_{23} \sin(w_2 - v_2),$$

and K_3 be the coefficient of cos w_2 in

$$\frac{h^2}{h_0^2} \cdot \frac{h_0^2}{u^2} \, C \, \lambda_{33} \sin(w_2 - v_2).$$

We have to put

$$(1 - g_2)(1 + \tau_1') + g_3(\cos i_0 + \tau_2') + K_1 = 0 \tag{50}$$

$$(1 - g_2)(\cos i_0 + \tau_2') - g_3(1 + \tau_1') + K_2 = 0 \tag{51}$$

$$(1 - g_2)(\sin i_0 + \tau_3') + K_3 = 0, \tag{52}$$

and because λ_{12} is of the order of perturbation, K_1 is of higher order relative to K_2 and K_3.

We determine $1-g_2$ and g_3 from equation (50) and (51). Equation (52) serves as check.

The constants c_1', c_2', c_3' in (50)-(52) can be taken from the previous approximation. It remains for us to discuss the determination of c_1'. Let $[\Gamma_1]$, $[\Gamma_2]$, $[\Gamma_3]$ be the values obtained by the formal process of integration, without adding the constants. Then:

$$\Gamma_1 = (1 + c_1') \cos w_2 + [\Gamma_1],$$

$$\Gamma_2 = (\cos i_0 + c_2') \sin w_2 + [\Gamma_2],$$

$$\Gamma_3 = (\sin i_0 + c_3') \sin w_2 + [\Gamma_3].$$

Let $[\Gamma_1]_0$, $[\Gamma_2]_0$, $[\Gamma_3]_0$ be the values of $[\Gamma_1]$, $[\Gamma_2]$, $[\Gamma_3]$ obtained for $w_2 = 0$.

$\vec{\Gamma}$ is a unit vector and it must be:

$$(1+c_1')^2 + \text{const. part in} \left\{ [\Gamma_1]_0^2 + [\Gamma_2]_0^2 + [\Gamma_3]_0^2 \right\} = 1$$

From this last equation $1+c_1'$ is obtained without any ambiguity. Determining constants in the given theory is simpler and easier than in the theory based on use of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Equations (11) can be written in the standard form

$$W = \Xi + \Upsilon \cos w_1 + \Psi \sin w_1 , \qquad (11')$$

where in our case

$$\Xi = \frac{h}{h_0} - \frac{h_0}{h} , \qquad (53)$$

$$\Upsilon = e \frac{h}{h_0} \cos\phi - \frac{h_0}{h} , \qquad (54)$$

$$\Psi = e \frac{h}{h_0} \sin\phi . \qquad (55)$$

If $[\Xi]$ $[\Upsilon]$ $[\Psi]$ and $[W]$ are the values obtained by formal integration, then

$$\Xi = [\Xi] + c_0'' ,$$

$$\Upsilon = [\Upsilon] + c_1'' ,$$

$$\Psi = [\Psi] ,$$

$$[W] = [\Xi] + [\Upsilon] \cos w_1 + [\Psi] \sin w_1 ,$$

$$W = [W] + c_0'' + c_1'' \cos w_1 ,$$

where c_0'' and c_1'' are the constants of integration.

We have

$[\Xi]$ = the part of $[W]$ which does not contain w_1,

$$[\Upsilon] = \{[W] - [\Xi]\}_{w_1 = 0}$$

$$[\Psi] = \{[W] - [\Xi]\}_{w_1 = \frac{\pi}{2}}$$

From (53) we obtain:

$$\frac{h_0^2}{h^2} = 1 - \Xi + \frac{1}{2}\Xi^2 - \frac{1}{8}\Xi^3 + \dots$$

$$\frac{h^2}{h_0^2} = 1 + \Xi + \frac{1}{2}\Xi^2 + \frac{1}{8}\Xi^3 + \dots \qquad (53')$$

$$\frac{h_0}{h} = 1 - \frac{1}{2}\Xi + \frac{1}{8}\Xi^2 + 0 \cdot \Xi^3 + \dots$$

$$\frac{h}{h_0} = 1 + \frac{1}{2}\Xi + \frac{1}{8}\Xi^2 + 0 \cdot \Xi^3 + \dots \qquad (53'')$$

where Ξ^2, Ξ^3, ... can be taken from the previous approximation.

We determine c_0'' in such a way that $\frac{h_0^2}{h^2} - 1$ does not contain a constant term. Then

$$c_0'' = - \text{ the constant term in } \left\{ + \frac{1}{2}\,\Xi^2 - \frac{1}{8}\,\Xi^3 + \cdots \right\},$$

c_0'' is of the higher order compared to Ξ. From (12) we obtain

$$\frac{u}{h^2} = \frac{h_0^2}{h^2} \cdot \frac{\bar{u}}{h_0^2} + \frac{h_0}{h}\,\overline{W}, \tag{56}$$

or

$$\frac{u}{h^2} = \frac{h_0^2}{h^2} \cdot \frac{\bar{u}}{h_0^2} + [\overline{W}] + c_0'' + c_1'' \cos v_1 + \left(\frac{h}{h_0} - 1 \right) \overline{W}, \tag{57}$$

$\frac{h}{h_0} - 1$ and $\overline{W}$ in the last equation can be taken from the previous approximation. We choose c_1'' in such a way that no term of the form A cos v_1 is present excepting the term $e_0 \cos v_1$.

Determination of Time

We introduce the "disturbed" time z by means of the equation

$$\frac{1}{\bar{u}^2} \cdot \frac{dv}{dz} = \frac{1}{h_0} \tag{58}$$

Introducing the eccentric anomaly ε by means of the equation

$$\bar{r} = \bar{u}^{-1} = h_0^{-2} (1 + e_0 \cos v_1)^{-1} = a_0 (1 - e_0 \cos \varepsilon)$$

where

$$a_0 = h_0^{-2} (1 - e_0^2)^{-1}$$

we can write (58) as a Kepler's equation of the form

$$\varepsilon - e_0 \sin \varepsilon = l_0 + n_0 g_1 z$$

$$n_0 = a_0^{-3/2}$$

Putting

$$z = t + \delta z$$

we obtain from (3), (12') and (58)

$$\frac{d\, n_0 \delta z}{dv} = (1 - e_0^2)^{3/2} \left(\frac{h_0^2}{\bar{u}}\right)^2 \left[1 - \frac{h}{h_0} (1 + \nu)^2\right]$$

Making use of (53") we obtain

$$\frac{d\, n_0 \delta z}{dv} = (1 - e_0^2)^{3/2} \left(\frac{h_0^2}{\bar{u}}\right)^2 \left[(2\nu + \frac{1}{2} \Xi) + (\frac{1}{8} \Xi^2 + \Xi \nu + \nu^2) + \dots\right]$$

Conclusion

The lunar theory presented here is a numerical one, based on the application of the process of iteration. The experimenting with Hansen's lunar theory done at Goddard Space Flight Center by M. Charnow and by the author confirms the possibility of using the process of iteration to solve the problem. The number of cycles for the satellites of outer planets will be rather small, because the final format of the computation accuracy for such satellites does not exceed 10^{-5} in $1 + \nu$ and 0.001° in the angles.

The decision concerning the choice of terms in the development is left to the machine and thus there is no danger that by accident some influential terms will be omitted. We assume that the eccentricity is small or moderate. Then validity of the theory depends upon the value of the parallactic factor and upon the presence of the resonance effects.

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